MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Problem Solving 3 Solutions: Calculating the Electric Field of Highly Symmetric Distributions of Charge Using Gauss's Law

REFERENCE: Section 4.2, 8.02 Course Notes.

Introduction

When approaching Gauss's Law problems, we described a problem solving strategy summarized below (see also, Section 4.7, 8.02 Course Notes):

Summary: Methodology for Applying Gauss's Law

Step 1: Identify the 'symmetry' properties of the charge distribution.

Step 2: Determine the direction of the electric field, and a surface on which the magnitude of the electric field is constant.

Step 3: Decide how many different regions of space the charge distribution determines

Step 4: For each region of space, choose the Gaussian surface such that the flux integral is either case a) or case b) above.

Step 5: Calculate the flux through the Gaussian surface.

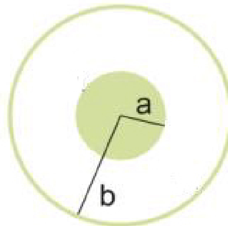
Step 6: For each region of space, calculate the charge enclosed in the choice of the Gaussian surface for that region.

Step 7: For each region of space, equate the two sides of Gauss's Law in order to find an expression for the magnitude of the electric field in that region of space.

Step 8: Graph the magnitude of the electric field as a function of the parameter specifying the Gaussian surface for all regions of space.

You should now apply this strategy to the following problem.

Question: ***Concentric Cylinders***



A long very thin non-conducting cylindrical shell of radius b and length L surrounds a long solid non-conducting cylinder of radius a and length L with $b > a$. The inner cylinder has a uniform distribution of charge $\rho = \dfrac{Q}{L\pi a^2}$ distributed throughout its volume, where $+Q$ is the total charge on the inner cylinder. On the outer cylinder we place an equal and opposite to charge, $-Q$. The region $a < r < b$ is empty.

Step 1 Question: (*Answer on the tear-sheet at the end!*) What is the 'symmetry' property of the charge distribution here?

Cylindrical

Step 2 Question: (*Answer on the tear-sheet at the end!*) What is the direction of the electric field, and what is a surface on which the magnitude of the electric field is constant?

The electric field is in the cylindrical radial direction, and its magnitude is constant on cylindrical surfaces with constant radius.

Step 3 Question: (*Put your answer on the tear-sheet at the end!*) How many different regions of space does the charge distribution determine?

Three: inside a, between a and b, and outside of b.

Step 4 Question: (*Put your answer on the tear-sheet at the end!*) For each region of space, describe your choice of the Gaussian surface. What variable did you choose to parameterize your Gaussian surface? What is the range of that variable?

In each region we take our Gaussian surface to be a cylinder of radius r and height h, co-axial with our real cylinders. We chose the variable r to parameterize our surface, and it ranges from 0 to ∞.

Step 5 Question: (*Put your answer on the tear-sheet at the end!*) For the region for $r < a$, calculate the flux through your choice of the Gaussian surface. Your expression should include the unknown electric field for that region.

We see that we will get no contribution to the flux from the ends (because the electric field and the area normal are perpendicular there), and over the sides of the Gaussian cylinder we have

$$\oiint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = 2\pi rhE$$

Step 6 Question: (*Put your answer on the tear-sheet at the end!*) For the region for $r < a$, calculate the charge enclosed in your choice of the Gaussian surface.

$$\frac{Q_{inside}}{\varepsilon_o} = \frac{1}{\varepsilon_o}\left[\pi r^2 h\right]\rho$$

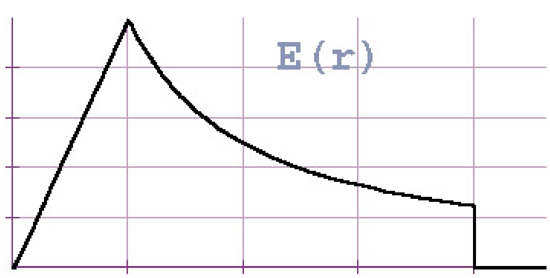
Step 7 Question 1: (*Put your answer on the tear-sheet at the end!*) For the region for $r < a$, equate the two sides of Gauss's Law that you calculated in steps 5 and 6, in order to find an expression for the magnitude of the electric field.

$$2\pi rhE = \frac{1}{\varepsilon_o}\left[\pi r^2 h\right]\rho, \quad \text{so } E_{r<a} = \frac{r\rho}{2\varepsilon_o} = \frac{Qr}{2\pi a^2 L\varepsilon_0}$$

Step 7 Question 2: (*Put your answer on the tear-sheet at the end!*) Repeat the same procedure in order to calculate the electric field as a function of r, the distance from the axis of the cylinders for the regions $a < r < b$.

$$\oiint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = 2\pi rhE = \frac{Q_{inside}}{\varepsilon_0} = \frac{Q}{\varepsilon_0}\frac{h}{L} \quad \Rightarrow \quad E(r)_{a<r<b} = \frac{Q}{2\pi rL\varepsilon_0}$$

Step 8 Question: (*Put your answer on the tear-sheet at the end!*) Make a graph in the space below of the magnitude of the electric field as a function of the parameter specifying the Gaussian surface for all regions of space.



Penultimate Question (*Put your answer on the tear-sheet at the end!*) What is the potential difference between $r = a$ and $r = 0$? That is, what is $\Delta V = V(a) - V(0)$?

First, lets find $V(r)$ everywhere. If $r > b$, and the potential is zero at infinity, then since there is no electric field between b and infinity, $V(r) = 0$. If $a < r < b$, then

$$V(r) = -\int_\infty^r \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\int_\infty^r E(r)\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}\, dr = -\int_\infty^r E(r) dr = -\int_b^r \frac{Q dr}{2\pi r L \varepsilon_o} = \frac{Q}{2\pi L \varepsilon_o} \ln \frac{b}{r}$$

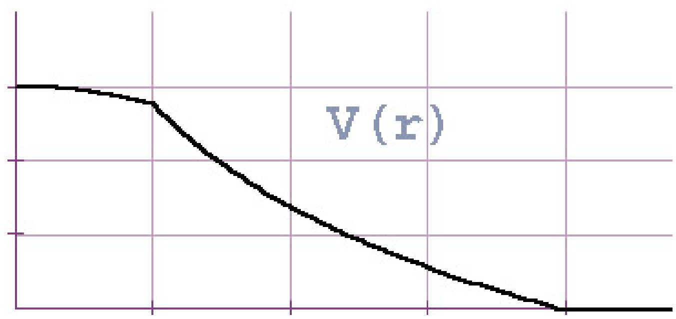
If $r < a$, then

$$V(r) = -\int_\infty^r \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\int_\infty^r E(r)\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}\, dr = -\int_\infty^r E(r) dr = -\int_b^a \frac{Q dr}{2\pi r L \varepsilon_o} - \int_a^r \frac{Q r dr}{2\pi a^2 L \varepsilon_o} =$$

$$\frac{Q}{2\pi L \varepsilon_o} \ln \frac{b}{a} - \frac{Q\left(r^2 - a^2\right)}{4\pi a^2 L \varepsilon_o} = \frac{Q}{2\pi L \varepsilon_o}\left[\ln \frac{b}{a} + \frac{1}{2}\left(1 - \frac{r^2}{a^2}\right)\right]$$

The potential is plotted below for all r. The potential increases by an amount $\frac{Q}{4\pi L \varepsilon_o}$ in going from a to 0, and $\Delta V = V(a) - V(0) = -\frac{Q}{4\pi L \varepsilon_o}$.

Note the sign here. We are asking an external agent to move a test charge from 0 to a, which is in the direction of the field. Thus that agent does negative work, since the field is actually doing work on the agent.



Final Question: (*Put your answer on the tear-sheet at the end!*) What is the potential difference between $r = b$ and $r = a$? That is, what is $\Delta V = V(b) - V(a)$?

Clearly, $\Delta V = V(b) - V(a) = -\frac{Q}{2\pi L \varepsilon_o} \ln \frac{b}{a}$. Note the sign here. We are asking an external agent to move a test charge from a to b, which is in the direction of the field. Thus that agent does negative work, since the field is actually doing work on the agent.